

Some finite properties for vertex operator superalgebras

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Abstract

Vertex operator superalgebras are studied and various results on rational Vertex operator superalgebras are obtained. In particular, the vertex operator super subalgebras generated by the weight $\frac{1}{2}$ and weight 1 subspaces are determined. It is also established that if the even part V_0 of a vertex operator superalgebra V is rational, so is V .

1. Introduction

The vertex operator superalgebras which are natural generalizations of vertex operator algebras have been studied extensively in [15], [16], [23], [29], [30] and [33]. In this paper, we study certain finite properties of vertex operator superalgebras following [10], [12], [13] and [32].

A vertex operator superalgebra $V = V_0 \oplus V_1$ has even part V_0 and odd part V_1 where V_0 consists of vectors of integral weights and V_1 consists of vectors whose weights are half integers but not integers. So there is a canonical automorphism σ of V acting on V_i as $(-1)^i$ and V_0 is a vertex operator algebra which is also a fixed point subalgebra of V . So a better understanding of the relationship between representation theories of V and V_0 is definitely useful for the study of orbifold theory (see [4] and [11]). Even the orbifold theory for vertex operator algebra with order 2 automorphism has not been understood fully.

Rationality which is an analogue of semisimplicity of associative and Lie algebras is probably the most important concept in the representation theory of vertex operator superalgebra. We first establish that if V_0 is rational then V is rational although we believe that the rationalities of V and V_0 are equivalent from the orbifold theory. The main tool is the associative algebras $A_{g,n}(V)$ for $n \in \frac{1}{2}\mathbb{Z}_+$ which are generalizations of $A_{g,n}(V)$ as introduced and studied in [10] (also see [34], [23], [8] and [9]) where g is an automorphism of V of finite order. It is established that V is g -rational if and only if $A_{g,n}$ is a finite dimensional semisimple associative algebra for large n . This is the key result to prove the rationality of V from the rationality of V_0 . Another characterization of rationality is given through Ext functor.

Our investigation next centers around the vertex operator super subalgebras of V generated by homogenous subspaces of small weights. The vertex operator subalgebra generated by $V_{\frac{1}{2}}$ is a holomorphic vertex operator superalgebra U associated to an infinite dimensional Clifford algebra built from a finite dimensional vector space with a nondegenerate symmetric bilinear form. This enables us to decompose V as a tensor product $U \otimes U^c$ where U^c whose weight $\frac{1}{2}$ subspace is zero is the commutant of U in V ([20], [27]). Moreover, the module

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categories of V and U^c are equivalent. To study V_1 we need to understand the algebraic structure of V_1 first. Under the assumptions that V is rational or σ -rational together with C_2 -cofiniteness we are able to show that V_1 is a reductive Lie algebra using the modular invariance results from [15] and [34] and the fact that $E_2(\tau)$ is not modular. Also the rank of V_1 and the dimension of $V_{\frac{1}{2}}$ are controlled by the effective central charge. Furthermore, for any simple Lie subalgebra \mathfrak{g} of V_1 , the vertex operator subalgebra generated by \mathfrak{g} is isomorphic to the vertex operator algebra $L(k, 0)$ which is the integrable highest weight module for the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$. We also give a rational vertex operator subalgebra which is a tensor product of affine vertex operator algebras and a lattice vertex operator algebra and whose weight one subspace is exactly the V_1 .

We should point out that most of results in this paper have been obtained in the case V is a vertex operator algebra in [10], [12], [13], [32]. So this paper can be regarded as a “super” analogues of results presented in [10], [12], [13], [32]. The main ideas and the broad outlines also follow from these papers. A lot of arguments are omitted if they are the same as in the case of vertex operator algebras. On the other hand, there is a new phenomenon in the super case. Namely, either rationality together with C_2 -cofiniteness or σ -rationality together with C_2 -cofiniteness implies that V_1 is reductive. This gives a strong evidence that rationality, σ -rationality of V and rationality of V_0 are equivalent. But we have no idea how to establish this.

This paper is organized as follows. We recall various notions of twisted modules for a vertex operator superalgebra and g -rationality for any automorphism of finite order from [19], [34], [8] and [16] in Section 2. In Section 3, we define a series of associative algebras $A_{g,n}(V)$ for a vertex operator superalgebra V and $n \in \mathbb{Z}_+$. We exhibit how to use $A_n(V)$ to prove rationality of V from the rationality of V_0 . It is also shown that if V is C_2 -cofinite or rational then V is finitely generated and the automorphism group $Aut(V)$ is an algebraic group. The Section 4 is devoted to the study of vertex operator super subalgebra generated by $V_{\frac{1}{2}}$. In Section 5 we show that if V is rational or σ -rational together with C_2 -cofiniteness then the weight one subspace V_1 is a reductive Lie algebra whose rank is bounded above by the effective central charge \tilde{c} . Consequently, $\dim V_{\frac{1}{2}}$ is bounded above by $2\tilde{c} + 1$. Section 6 deals with the vertex operator subalgebra of V generated by V_1 and related.

We make the assumption that the reader is familiar with the theory of vertex operator algebras as presented in [3], [6], [19] and [27].

2. Basics

In this section we give the definition of vertex operator superalgebra and several notions of modules (cf. [7], [16] [17], [19], [29], [34]).

We first recall that a super vector space is \mathbb{Z}_2 -graded vector space $V = V_0 \oplus V_1$. The elements in V_0 (resp., V_1) are called even (resp., odd). Let \tilde{v} be 0 if $v \in V_0$, and 1 if $v \in V_1$.

Definition 2.1 *A vertex operator superalgebra (VOSA) is a $\frac{1}{2}\mathbb{Z}$ -graded vector space*

$$V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n = V_0 \oplus V_1$$

with $V_0 = \sum_{n \in \mathbb{Z}} V_n$ and $V_1 = \sum_{n \in \mathbb{Z}} V_{n+\frac{1}{2}}$ satisfying all the axioms in the definition of vertex

operator algebra except that the Jacobi identity is replaced by:

$$\begin{aligned} & z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1) Y(v_2, z) - (-1)^{\tilde{u}\tilde{v}} z_0^{-1} \delta\left(\frac{-z_2 + z_1}{z_0}\right) Y(v, z_2) Y(u, z_1) \\ &= z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2). \end{aligned}$$

Throughout the paper we always assume that V is a vertex operator superalgebra unless otherwise stated.

Definition 2.2 *An automorphism g of a VOSA V is a linear automorphism of V preserving the vacuum vector $\mathbf{1}$ and the conformal vector ω such that the actions of g and $Y(v, z)$ on V are compatible in the sense that*

$$gY(v, z)g^{-1} = Y(gv, z)$$

for $v \in V$.

Denote by $Aut(V)$ the set consisting of all automorphisms of V . Observe that any automorphism of V commutes with $L(0)$ and hence preserves each homogeneous subspace V_n . As a consequence, any automorphism preserves both $V_{\bar{0}}$ and $V_{\bar{1}}$. There is a canonical automorphism σ of V with $\sigma|_{V_i} = (-1)^i$ associated to the \mathbb{Z}_2 -grading of V .

Let $g \in Aut(V)$ with finite order T , then we can decompose V into eigenspaces of g :

$$V = \bigoplus_{r=0}^{T-1} V^r$$

where $V^r = \{v \in V | gv = e^{\frac{-2\pi ir}{T}} v\}$.

Definition 2.3 *A weak g -twisted V -module M is a \mathbb{Z}_2 -graded vector space equipped with a linear map*

$$\begin{aligned} & V \rightarrow (\text{End} M)[[z, z^{-1}]], \\ & v \mapsto Y_M(v, z) = \sum_{n \in \frac{1}{T}\mathbb{Z}} v_n z^{-n-1}, \end{aligned}$$

such that for all $u \in V^r$ ($0 \leq r \leq T-1$), $v \in V$ and $w \in W$ the following conditions hold:

$$\begin{aligned} & Y_M(u, z) = \sum_{n \in \frac{r}{T} + \mathbb{Z}} u_n z^{-n-1}; \\ & u_n w = 0, \quad \text{for } n \gg 0; \\ & Y_M(1, z) = \text{Id}_M; \end{aligned}$$

$$\begin{aligned} & z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u, z_1) Y_M(v, z_2) - (-1)^{\tilde{u}\tilde{v}} z_0^{-1} \delta\left(\frac{-z_2 + z_1}{z_0}\right) Y_M(v, z_2) Y_M(u, z_1) \\ &= z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right)^{-r/T} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u, z_0)v, z_2). \end{aligned}$$

Definition 2.4 An admissible g -twisted V -module is a weak g -twisted V -module M which carries a $\frac{1}{T}\mathbb{Z}_+$ -grading

$$M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)$$

satisfying

$$v_m M(n) \subseteq M(n + \text{wt} v - m - 1)$$

for homogeneous $v \in V$.

Definition 2.5 An ordinary g -twisted V -module is a weak g -twisted V -module

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$$

such that $\dim M_\lambda$ is finite and for fixed λ , $M_{\frac{n}{T}+\lambda} = 0$ for all small enough integers n where $M_\lambda = \{w \in M | L(0)w = \lambda w\}$ and $Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$.

We say V is g -rational if every admissible g -twisted V -module is completely reducible, i.e. a direct sum of simple admissible g -twisted V -modules. V is g -regular if the category of weak g -twisted V -modules is semisimple, namely, every weak g -twisted V -module is a direct sum of irreducible weak g -twisted V -modules. If $g = 1$, we have the definitions of rationality and regularity for vertex operator superalgebras.

The following definitions are given for vertex operator algebras in [13] and [34] and we extend these to vertex operator superalgebras here.

A vertex operator superalgebra V is said to be of *CFT type* in case that the $L(0)$ -grading on V has no negative weights, and if the degree-zero homogeneous subspace V_0 is 1-dimensional, i.e. $V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} V_n$ and $V_0 = \mathbb{C}\mathbf{1}$. We say V is of *strong CFT type* if V satisfies the further condition $L(1)V_1 = 0$. V is said to be *C_2 -cofinite* in case that $C_2(V)$ has finite codimension in V , where $C_2(V)$ is the subspace of V linearly spanned by all elements of the form $u_{-2}v$ for $u, v \in V$.

For convenience, let us introduce the term *strongly g -rational* for a simple vertex operator superalgebra V which satisfies the following conditions:

- (1) V is of strong CFT type;
- (2) V is C_2 -cofinite;
- (3) V is g -rational.

Definition 2.6 A bilinear form (\cdot, \cdot) on V -module M is said to be invariant [18] if it satisfies the following condition

$$(Y(a, z)u, v) = (u, Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})v) \quad \text{for } a \in V, u, v \in M.$$

It is proved in [28] and [33] that there exists a linear isomorphism from the space of invariant bilinear forms on V to $\text{Hom}_{\mathbb{C}}(V_0/L(1)V_1, \mathbb{C})$. This implies that there is a unique, up to multiplication by a nonzero scalar, nondegenerate symmetric invariant bilinear form on V if V is simple and of strong CFT type.

3. Rationality

In this section we give a characterization of rationality of a vertex operator superalgebra V in terms of rationality of vertex operator subalgebra $V_{\bar{0}}$. We will show that if $V_{\bar{0}}$ is rational then V is rational. We certainly believe that the converse is also true. That is, if V is rational then $V_{\bar{0}}$ is also rational. This is similar to a well-known conjecture in orbifold theory: Let V be a rational vertex operator algebra and g is an order 2 automorphism of V . Then the fixed point vertex operator subalgebra is also rational. We will establish some other results on rationality. We also discuss the generators of V .

The tool we use to prove the main result is the associative algebras $A_n(V)$ which is defined in [9] for vertex operator algebra. Let V be vertex operator superalgebra. Let $O_n(V)$ be the subspace of V linearly spanned by all $L(-1)u + L(0)u$ and $u \circ_n v$ where for homogeneous $u \in V$ and $v \in V$

$$u \circ_n v = \begin{cases} \text{Res}_z \frac{(1+z)^{\text{wt}u+n}}{z^{2n+2}} Y(u, z)v, & \text{if } u \in V_{\bar{0}}, \\ \text{Res}_z \frac{(1+z)^{\text{wt}u+n-\frac{1}{2}}}{z^{2n+1}} Y(u, z)v, & \text{if } u \in V_{\bar{1}}. \end{cases}$$

Define another operation $*_n$ on V by

$$u *_n v = \begin{cases} \sum_{m=0}^n (-1)^m \binom{m+n}{n} \text{Res}_z \frac{(1+z)^{\text{wt}u+n}}{z^{n+m+1}} Y(u, z)v, & \text{if } u, v \in V_{\bar{0}}, \\ 0, & \text{if } u \in V_{\bar{1}} \text{ or } v \in V_{\bar{1}}. \end{cases}$$

Set $A_n(V) = V/O_n(V)$. Then $A_0(V)$ is the $A(V)$ studied in [23]. Let M be a weak V -module. Define the “ n -th lowest weight vector” subspace of M to be

$$\Omega_n(M) = \{w \in M \mid u_{\text{wt}u+n+i} w = 0, u \in V, i \geq 0\}.$$

As in [9] we have the following results:

Theorem 3.1 (1) Suppose that M is a weak V -module. Then $\Omega_n(M)$ is an $A_n(V)$ -module such that a acts as $o(a)$ for $a \in V_{\bar{0}}$, where $o(a)$ is defined to be $a_{\text{wt}a-1}$ for homogeneous $a \in V_{\bar{0}}$ and extends it linearly.

(2) Suppose that $M = \oplus_{i \in \frac{1}{2}\mathbb{Z}_+} M(i)$ is an admissible V -module. Then (a) $\Omega_n(M) \supset \oplus_{i \leq n} M(i)$; (b) Assume that M is simple. Then $\Omega_n(M) = \oplus_{i \leq n} M(i)$, and each $M(i)$ is a simple $A_n(V)$ -module for $i = 0, \frac{1}{2}, \dots, n$.

(3) $M \mapsto M(0)$ gives a bijection between irreducible admissible V -modules and simple $A(V)$ -modules

(4) The identity map induces an epimorphism from $A_n(V)$ to $A_m(V)$ for any $n \geq m$.

(5) If V is g -rational there are only finitely many irreducible admissible g -twisted V -modules up to isomorphism and each irreducible admissible g -twisted V -module is ordinary.

We should point out that part (3) of the theorem was obtained in [23].

The next lemma will be used as a characterization of rationality of V in terms of semisimplicity of $A_n(V)$ for large enough n .

Lemma 3.2 Suppose that $A(V)$ is finite dimensional, then any admissible V -module is a direct sum of generalized eigenspaces for $L(0)$.

Proof. Let $M = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}_+} M(i)$ be an admissible V -module with $M(0) \neq 0$. Let W be a maximal subspace of M which is a direct sum of generalized eigenspaces with respect to $L(0)$. Then it is not hard to see that W is a submodule of M . Consider the $A(V)$ -module $M(0)$. By our assumption on finite dimension of $A(V)$ we see that there exists a nonzero simple $A(V)$ -submodule of $M(0)$, on which $L(0)$ acts as a scalar by Schur lemma. This shows that $W \neq 0$. We shall show $W = M$. Suppose $M/W \neq 0$. Choose the minimal $n \in \frac{1}{2}\mathbb{Z}_+$ such that $M(n)/W(n) \neq 0$, where $W(n) = W \cap M(n)$. Then by the similar argument as above, we see that $M(n)/W(n)$ contains a nonzero simple $A(V)$ -submodule, say $\mathcal{W}(n)/W(n) \neq 0$, where $\mathcal{W}(n)$ is a subspace of $M(n)$. Since both $\mathcal{W}(n)/W(n)$ and $W(n)$ are a direct sum of generalized eigenspaces for $L(0)$, then so is $\mathcal{W}(n)$. Thus $\mathcal{W}(n) \subset W$ and $\mathcal{W}(n) = W(n)$, a contradiction. \square

Assume that $A(V)$ is finite dimensional. Let $f(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_r) \in \mathbb{C}[x]$ be the monic polynomial of least degree such that $f([w]) = 0$ in $A(V)$. Then on any given simple $A(V)$ -module $L(0)$ must act as a constant λ_i for some i . Note from Theorem 3.1 that V has exactly r inequivalent irreducible admissible modules $M^i = \sum_{n \in \frac{1}{2}\mathbb{Z}_+} M_{\lambda_i+n}^i$ for $i = 1, \dots, r$. Then there exists $m_i > 0$ such that $M_{\lambda_i+n}^i \neq 0$ for all $n \geq m_i$. Let N be a positive integer greater than $|\lambda_i - \lambda_j|$, $|\lambda_i| + 1$, and m_i for $i, j = 1, \dots, r$.

Note that the rationality is defined from the representation theory. It is always believed that such property, which is analogous to the semisimplicity of Lie and associative algebras, should have its own internal characterization. The the following result can be regarded as an internal characterization of rationality.

Theorem 3.3 *V is rational if and only if $A_n(V)$ is finite dimensional and semisimple for some $n \geq N$.*

Proof. The proof of Theorem 4.10 in [9] shows that V is rational then $A_n(V)$ is semisimple and finite dimensional for all n . Now we assume that $A_n(V)$ is semisimple for some $n \geq N$. By Theorem 3.1 $A_m(V)$ is semisimple for all $m \leq n$. Let $M = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}_+} M(i)$ be an admissible V -module with $M(0) \neq 0$. By Lemma 3.2, we can write

$$M = \sum_{\lambda \in \{\lambda_1, \dots, \lambda_r\}} \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} M_{\lambda+n}$$

where $M_{\lambda+n}$ is the generalized eigenspace for $L(0)$ with eigenvalue $\lambda + n$. Note that for each $\lambda \in \{\lambda_1, \dots, \lambda_r\}$ the subspace $\bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} M_{\lambda+n}$ is an admissible submodule of M . Without loss of generality, we may assume that $M = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} M(n)$ for some $\lambda \in \{\lambda_1, \dots, \lambda_r\}$ where $M(n) = M_{\lambda+n}$.

We assert that the submodule W generated by $\bigoplus_{n \leq N, n \in \frac{1}{2}\mathbb{Z}_+} M(n)$ is equal to the entire M . Otherwise, $0 \neq M/W = \bigoplus_{n > N, n \in \frac{1}{2}\mathbb{Z}_+} M(n)/W(n)$ where $W(n) = W \cap M(n)$. Let $n_0 \in \frac{1}{2}\mathbb{Z}_+$ be minimal such that $M(n_0)/W(n_0) \neq 0$. Then $n_0 > N$ and $M(n_0)/W(n_0)$ is an $A(V)$ -module by Theorem 3.1. Since $A(V)$ is semisimple, there exists a nonzero simple $A(V)$ -submodule of $M(n_0)/W(n_0)$ on which $L(0)$ acts as the constant $\lambda + n_0 \in \{\lambda_1, \dots, \lambda_r\}$, which implies $|\lambda - \lambda_j| = n_0$ for some j . But this is impossible by our choice on N . Thus we must have $W = M$.

We next show that if X is a simple $A(V)$ -submodule then X generates an irreducible V -module U . Denote $J = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} J(n)$ the maximal submodule of U such that $J(0) = 0$,

where $J(n) = J \cap U(n)$. Then the quotient $W = U/J$ is irreducible and $W(0) = X$. Since $\bigoplus_{0 \leq n \leq N} U(n)$ is a semisimple $A_N(V)$ -module we can regard each $W(n)$ a $A_N(V)$ -submodule of $U(n)$ for $n \leq N$. From the choice of N we know that $W(N) \neq 0$. Then the admissible V -submodule of U generated by $W(N)$ contains $W(0) = X$. Thus $W(N) = U(N)$ and therefore $J(N) = 0$. By our choice of N again we see that J must be trivial. This implies that $U = W$ is irreducible.

It follows that the admissible V -submodule \mathcal{W} of M generated by $M(0)$ is completely reducible. Note that $M(1) = \mathcal{W}(1) \oplus P$ where P is a semisimple $A(V)$ -module. Again the admissible submodule of M generated by P is completely reducible. Continuing in this way completes the proof. \square

Remark 3.4 *Even in the case that V is a vertex operator algebra, Theorem 3.3 strengthens the Theorem 4.11 of [9] where we require that $A_n(V)$ is semisimple for all n .*

Remark 3.5 *There is a twisted analogue $A_{g,n}(V)$ (cf. [10]) of $A_n(V)$. One can similarly define the positive integer N_g . Then Theorem 3.3 still holds. That is, V is g -rational if and only if $A_{g,n}(V)$ is finite dimensional and semisimple for some $n \geq N_g$.*

We now use Theorem 3.3 to prove the following result:

Proposition 3.6 *Let $V = V_0 \oplus V_1$ be a VOSA. If V_0 is rational, then V is rational.*

Proof. Suppose that V_0 is rational. Then by Theorem 3.3, $A_n(V_0)$ is finite dimensional semisimple associative algebra if n is sufficiently large. This implies that $A_n(V)$ is semisimple as $A_n(V)$ is a quotient of $A_n(V_0)$. Applying Theorem 3.3 again yields that V is rational. \square

We remark that we do not know how to prove the rationality of V from the rationality of V_0 without using $A_n(V)$. It is certainly a very interesting problem to find a different approach without using $A_n(V)$. Although we can not show the converse of Proposition 3.6 we strongly believe that rationalities of V and V_0 are equivalent.

In the rest of this section we use the extension functor to consider the rationality of a vertex operator superalgebra V . This approach has been studied in [1] for vertex operator algebra. But our rationality result is different from that given in [1].

First let us describe the set $\text{Ext}_V^1(M^2, M^1)$ for any weak V -module M^1 and M^2 . We call a weak V -module M an *extension of M^2 by M^1* if there is a short exact sequence $0 \rightarrow M^1 \rightarrow M \rightarrow M^2 \rightarrow 0$. Two extensions M and N of M^2 by M^1 are said to be equivalent if there exists a V -homomorphism $f : M \rightarrow N$ such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^1 & \xrightarrow{\phi} & M & \xrightarrow{\varphi} & M^2 \longrightarrow 0 \quad (\text{exact}) \\ & & \parallel & & \downarrow f & & \parallel \\ 0 & \longrightarrow & M^1 & \xrightarrow{\phi'} & N & \xrightarrow{\varphi'} & M^2 \longrightarrow 0 \quad (\text{exact}). \end{array}$$

Define $\text{Ext}_V^1(M^2, M^1)$ to be the set of all equivalent classes of M^2 by M^1 . It is well known that $\text{Ext}_V^1(M^2, M^1)$ carries the structure of an abelian group such that the equivalent class of $M^1 \oplus M^2$ is the zero element.

Here is another equivalent condition of rationality.

Proposition 3.7 *Let V be a vertex operator superalgebra. Then V is rational if and only if the following two conditions hold:*

- (a) *Every admissible V -module contains a nontrivial irreducible admissible submodule;*
- (b) *For any irreducible V -modules M and N , $\text{Ext}_V^1(M, N) = 0$.*

Proof. It is clear that rationality implies both (a) and (b). Now we assume that (a) and (b) hold. Let $M = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} M(n)$ be a nonzero admissible V -module. Let W be the sum of irreducible admissible V -submodules of M . Then $W = \bigoplus_{i \in I} W^i$ where each W^i is an irreducible admissible V -module. By condition (a), $W \neq 0$. We assert that $W = M$. Otherwise consider the quotient module M/W . It follows from the condition (a) again that there exists a weak V -submodule M' such that $M' \not\supseteq W$ and M'/W is an irreducible admissible V -module. Then by condition (b) and the properties of Ext we have

$$\text{Ext}_V^1(M'/W, W) = \bigoplus_{i \in I} \text{Ext}_V(M'/W, W^i) = 0.$$

That is, $M' = M'/W \oplus W$ as V -modules, contradicting the maximality of W . So the assertion is true and M is a direct sum of irreducible admissible V -modules. \square

We now turn our attention to the generators of vertex operator superalgebras.

Proposition 3.8 *Let V be vertex operator superalgebra. Then we have*

- (a) *If V is rational or C_2 -cofinite, then V is finitely generated.*
- (b) *If V is finitely generated, then $\text{Aut}(V)$ is an algebraic group.*

The proof of these results have been given in the case of vertex operator algebras in [14], [24] (also see [21]) and [5]. The same proof works here.

4. Vertex operator subalgebra generated by $V_{\frac{1}{2}}$

In this section we study the vertex operator super subalgebra U of V generated by $V_{\frac{1}{2}}$ and decompose V as a tensor product $U \otimes U^c$ where U is holomorphic in the sense that U is the only irreducible module for itself and U^c whose weight $\frac{1}{2}$ subspace is 0 is the commutant of U in V . This decomposition reduces the study of vertex operator superalgebras to the study of vertex operator superalgebras whose weight $\frac{1}{2}$ subspaces are 0.

Let V be a simple vertex operator superalgebra of strong CFT type. Then there is a unique invariant, symmetric and nondegenerate bilinear form (\cdot, \cdot) such that

$$(\mathbf{1}, \mathbf{1}) = \sqrt{-1} \tag{4.1}$$

(see [28] and [33]). Then for $u, v \in V_{\frac{1}{2}}$ one has

$$u_0 v = (u, v) \mathbf{1} \tag{4.2}$$

and

$$[u(m), v(n)]_+ = (u, v) \delta_{m+n+1, 0}. \tag{4.3}$$

Note that the restriction of (\cdot, \cdot) to $V_{\frac{1}{2}}$ is still nondegenerate. Let $\{a^1, a^2, \dots, a^l\}$ be an orthonormal basis of $V_{\frac{1}{2}}$ with respect to the form (\cdot, \cdot) where $l = \dim V_{\frac{1}{2}}$.

Let U the vertex super subalgebra of V generated by $V_{\frac{1}{2}}$. Then using the equation (4.3) we see that

$$U = \text{Span}\{u_{-n_1}^1 u_{-n_2}^2 \cdots u_{-n_r}^r \mathbf{1} \mid u^i \in V_{\frac{1}{2}}, n_1 \geq n_2 \geq \cdots \geq n_r > 0 \text{ and } r \in \mathbb{Z}_+\}.$$

In fact, U carries the structure of a vertex operator superalgebra with conformal vector

$$\omega' = \frac{1}{2} \sum_{i=1}^l a_{-2}^i a_{-1}^i \mathbf{1}.$$

Define operators $L'(n)$ for $n \in \mathbb{Z}$ by:

$$Y(\omega', z) = \sum_{n \in \mathbb{Z}} w'_n z^{-n-1} = \sum_{n \in \mathbb{Z}} L'(n) z^{-n-2}.$$

Then the weight n subspace U_n for $L'(0)$ is given as follows

$$U_n = \left\langle u_{-n_1}^1 u_{-n_2}^2 \cdots u_{-n_r}^r \mathbf{1} \mid \begin{array}{l} u^i \in V_{\frac{1}{2}}, n_1 \geq n_2 \geq \cdots \geq n_r > 0, r \in \mathbb{Z}_+, \\ \text{and } n_1 + n_2 \cdots + n_r = n + \frac{r}{2} \end{array} \right\rangle.$$

It is well known (cf. [23]) that the vertex operator algebra U generated by $V_{\frac{1}{2}}$ is holomorphic. So for any admissible V -module M , we can decompose M into irreducible U -modules as follows

$$M = U \otimes \bar{M},$$

where $\bar{M} = \{w \in M \mid u_n w = 0 \text{ for all } u \in U \text{ and } n \in \mathbb{Z}_+\}$ is the multiplicity space of U in M . If $M = V$ the multiplicity space \bar{M} is denoted by U^c and is called the commutant of U in V . In particular, $V = U \otimes U^c$. The U^c is a vertex operator superalgebra (see [20] and [27]) with $\omega - \omega'$ as its conformal vector and $U_{\frac{1}{2}}^c = 0$.

Let $\text{Irr}(V)$ and $\text{Irr}(U^c)$ denote the sets of the isomorphism classes of admissible irreducible V -modules and U^c -modules, respectively. The following result is straightforward.

Proposition 4.1 *Let V be a vertex operator superalgebra. Then*

- (a) *For any admissible V -module M , \bar{M} is an admissible U^c -module. Moreover, M is irreducible if and only if \bar{M} is irreducible;*
- (b) *The map $U \otimes * : \text{Irr}(U^c) \rightarrow \text{Irr}(V)$ is a bijection.*
- (c) *V is rational if and only if U^c is rational.*

5. The structure of weight 1 subspace

In this section we will investigate the Lie algebra structure of weight 1 subspace V_1 and show that V_1 is a reductive Lie algebra if V is σ -rational using the modular invariance results obtained in [16]. We also find an upper bound for the rank of V_1 in terms of effective central charge. Similar results for vertex operator algebra were given previously in [12] and the proof presented here is a modification of that used in [12]. We also apply these results to estimate the dimension of weight $\frac{1}{2}$ subspace $V_{\frac{1}{2}}$ of V .

We need some discussion on vertex operator superalgebra on torus ([34] and [15]), vector-valued modular forms [26] and the modular invariance of trace functions ([34] and [15]).

Let V be a vertex operator superalgebra. The vertex operator superalgebra

$$(V, Y[v, z], \mathbf{1}, \tilde{\omega})$$

on torus (see [34] and [15]) is defined as follows:

$$Y[v, z] = Y(v, e^z - 1)e^{z\text{wt}v} = \sum_{n \in \mathbb{Z}} v[n]z^{-n-1}$$

$$Y[\tilde{w}, z] = \sum_{n \in \mathbb{Z}} L[n]z^{-n-2}$$

for homogeneous v and $\tilde{\omega} = \omega - \frac{c}{24}$.

We denote the eigenspace of $L[0]$ with eigenvalue $n \in \frac{1}{2}\mathbb{Z}$ by $V_{[n]}$. If $v \in V_{[n]}$ we write $\text{wt}[v] = n$.

A holomorphic *vector-valued modular form of weight k* (k any real number) on the modular group $\Gamma = SL(2, \mathbb{Z})$ may be described as follows: for any integer $p \geq 1$ it is a tuple $(T_1(\tau), \dots, T_p(\tau))$ of functions holomorphic in the complex upper half-plane \mathbf{H} together with a p -dimensional complex representation $\rho : \Gamma \rightarrow \text{GL}(p, \mathbb{C})$ satisfying the following conditions:

(a) For all $\gamma \in \Gamma$ we have

$$(T_1, \dots, T_p)^t|_k \gamma(\tau) = \rho(\gamma)(T_1(\tau), \dots, T_p(\tau))^t$$

(t refers to transpose of vectors and matrices).

(b) Each function $T_j(\tau)$ has a convergent q -expansion holomorphic at infinity:

$$T_j(\tau) = \sum_{n \geq 0} a_n(j)q^{n/N_j}$$

for positive integer N_j . (Here and below, $q = \exp 2\pi i\tau$).

The following result [26] plays an important role in this section.

Proposition 5.1 *Let (T_1, \dots, T_p) be a holomorphic vector-valued modular form of weight k associated to a representation ρ of Γ . Then there is a nonnegative constant α depending only on ρ such that the Fourier coefficients $a_n(j)$ satisfy the polynomial growth condition $a_n(j) = O(n^{k+2\alpha})$ for every $1 \leq j \leq p$.*

Fix automorphisms g, h of V of finite orders. Let M be a simple $g\sigma$ -twisted V -module. Then $M = \bigoplus_{n=0}^{\infty} M_{\lambda + \frac{n}{T'}}$ for some λ called the conformal weight of M ($M_\lambda \neq 0$), where T' is the order of $g\sigma$. Suppose that M is σh -stable, which is equivalent to the existence of a linearly isomorphic map $\phi(\sigma h) : M \rightarrow M$ such that

$$\phi(\sigma h)Y_M(v, z)\phi(\sigma h)^{-1} = Y_M((\sigma h)v, z)$$

for all $v \in V$. From now on we assume that V is C_2 -cofinite. Then function F_M which is linear in $v \in V$ is defined for homogenous $v \in V$ as follows:

$$F_M(v, \tau) = q^{\lambda - c/24} \sum_{n=0}^{\infty} \text{tr}_{M_{\lambda + \frac{n}{T'}}} o(v)\phi(\sigma h)q^{\frac{n}{T'}} = \text{tr}_M o(v)\phi(\sigma h)q^{L(0) - c/24} \quad (5.1)$$

which is a holomorphic function in the upper half plane \mathbf{H} [15]. Here and below we write $F_M(\tau)$ rather than $F_M(\mathbf{1}, \tau)$ for simplicity. Then for any u, v in V such that $gv = hv = v$ we have

$$\mathrm{tr}_M o(u)o(v)\phi(\sigma h)q^{L(0)-c/24} = F_M(u[-1]v, \tau) - \sum_{k \geq 1} E_{2k}(q)F_M(u[2k-1]v, \tau) \quad (5.2)$$

(see [15] and [34]). The functions $E_{2k}(\tau)$ are the Eisenstein series of weight $2k$:

$$E_{2k}(q) = \frac{-B_{2k}}{2k!} + \frac{2}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n$$

where $\sigma_k(n)$ is the sum of the k -powers of the divisors of n and B_{2k} is a Bernoulli number. The $E_2(\tau)$ enjoys an exceptional transformation law. Namely, its transformation with respect to the matrix $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as follows:

$$E_2\left(\frac{-1}{\tau}\right) = \tau^2 E_2(\tau) - \frac{\tau}{2\pi i}. \quad (5.3)$$

We also need results on 1-point functions on torus from [15]. Let g, h be automorphisms of V of finite orders. The space of (g, h) 1-point functions $\mathcal{C}(g, h)$ is the \mathbb{C} -linear space consisting of functions

$$S : V \times \mathbf{H} \rightarrow \mathbb{C}$$

satisfying certain conditions (see [15] for details). The following results can be found in [15].

Theorem 5.2 *Let V be C_2 -cofinite and $g, h \in \mathrm{Aut}(V)$ of finite orders. Then (1) For $S \in \mathcal{C}(g, h)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we define*

$$S|_{\gamma}(v, \tau) = S|_k(v, \tau) = (c\tau + d)^{-k} S(v, \tau)$$

for $v \in V_{[k]}$, and extend linearly. Then $S|_{\gamma} \in \mathcal{C}((g, h)\gamma)$, where $(g, h)\gamma = (g^a h^c, g^b h^d)$.

(2) Let M be a simple $g\sigma$ -twisted V -module such that M is h and σ -stable. Then $F_M(v, \tau) \in \mathcal{C}(g, h)$.

(3) Suppose that V is $g\sigma$ -rational and M^1, \dots, M^m are the inequivalent, simple $g\sigma$ -twisted V -module such that M^i is h and σ -stable. Let F_1, \dots, F_m be the corresponding trace functions defined by (5.1). Then F_1, \dots, F_m form a basis of $\mathcal{C}(g, h)$.

We now assume that V is of strong CFT type. Recall from [18] that the weight 1 subspace V_1 of V carries a natural Lie algebra structure, the Lie bracket being given by $[u, v] = u_0 v$ for $u, v \in V_1$. Then any weak V -module is automatically a V_1 -module such that $v \in V_1$ acts as v_0 . Note that there is a unique nondegenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ such that $\langle \mathbf{1}, \mathbf{1} \rangle = -1$ and the restriction of $\langle \cdot, \cdot \rangle$ to V_1 endows V_1 with a nondegenerate, symmetric, invariant bilinear form such that $u_1 v = \langle u, v \rangle \mathbf{1}$ for $u, v \in V_1$.

The following two theorems are extensions of similar results from vertex operator algebras [12] to vertex operator superalgebras.

Theorem 5.3 *Let V be a strongly rational or strongly σ -rational. Then the Lie algebra V_1 is reductive.*

Proof. We first deal with the case that V is σ -rational. We have to show that the nilpotent radical N of the Lie algebra V_1 is zero. Suppose not, take any nonzero element $u \in N$. Each V_i for $i \in \frac{1}{2}\mathbb{Z}$ is finite dimensional V_1 -module and has a composition series $0 = W^0 \subset W^1 \subset W^2 \subset W^3 \subset \dots$ such that u_0 acts trivially on each composite factor W^i/W^{i-1} ($i = 1, 2, \dots$). Note that we can take $\phi(\sigma) = \sigma$ on V . Thus V is σ -stable. In fact, any irreducible V -module is σ -stable (see Lemma 6.1 of [15]). As a result, $\text{tr}_{V_i} o(u) o(v) \sigma = 0$ for all $v \in V_1$ and $i \in \frac{1}{2}\mathbb{Z}$. It follows from (5.2) that

$$F_V(u[-1]v, \tau) = \sum_{k \geq 1} E_{2k}(\tau) F_V(u[2k-1]v, \tau) \quad (5.4)$$

where $(g, h) = (\sigma, 1)$ and $F_V \in \mathcal{C}(\sigma, 1)$ by Theorem 5.2.

Note that if $k > 1$ is an integer, then the element $u[2k-1]v$ has $L[0]$ -weight $2 - 2k < 0$ and hence is 0. The non-degeneracy of the bilinear form $\langle \cdot, \cdot \rangle$ guarantees that there exists $v \in V_1$ such that $\langle u, v \rangle = 1$. With this choice of v , (5.4) simplifies to read

$$F_V(u[-1]v, \tau) = E_2(\tau) F_V(\tau). \quad (5.5)$$

By Theorem 3.1, V has finitely many irreducible σ -twisted V -modules up to isomorphism. We denote these modules by M^1, \dots, M^m . Note from Theorem 5.2 that the $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma$ maps $\mathcal{C}(\sigma, 1)$ to $\mathcal{C}(1, \sigma)$. By Theorem 5.2 again we see that

$$F_V(u[-1]v, -\frac{1}{\tau}) = \tau^2 \sum_{i=1}^m s_i F_{M^i}(u[-1]v, \tau)$$

and

$$F_V(-\frac{1}{\tau}) = \sum_{i=1}^m s_i F_{M^i}(\tau)$$

for some $s_i \in \mathbb{C}$. Similar to equality (5.5) we also have

$$F_{M^i}(u[-1]v, \tau) = E_2(\tau) F_{M^i}(\tau) \quad \text{for } i = 1, \dots, m.$$

Thus

$$\begin{aligned} \tau^2 \sum_{i=1}^m s_i F_{M^i}(u[-1]v, \tau) &= F_V(u[-1]v, -\frac{1}{\tau}) \\ &= E_2(-\frac{1}{\tau}) F_V(-\frac{1}{\tau}) \\ &= (\tau^2 E_2(\tau) - \frac{\tau}{2\pi i}) \sum_{i=1}^m s_i F_{M^i}(\tau) \\ &= \tau^2 \sum_{i=1}^m s_i F_{M^i}(u[-1]v, \tau) - \frac{\tau}{2\pi i} \sum_{i=1}^m s_i F_{M^i}(\tau). \end{aligned}$$

Of course, the equality (5.3) is involved in the calculations above. Canceling the term $\tau^2 \sum_{i=1}^m s_i F_{M^i}(u[-1]v, \tau)$ gives rise to the identity $\sum_{i=1}^m s_i F_{M^i}(\tau) = 0$, which in turn implies $F_V(\frac{-1}{\tau}) = 0$. But this is clearly not true, since

$$F_V(\frac{-1}{\tau}) = q^{-\frac{c}{24}} \left(\sum_{n \in \mathbb{Z}} (\dim V_n) q^n - \sum_{n \in \frac{1}{2} + \mathbb{Z}} (\dim V_n) q^n \right) \neq 0.$$

So $N = 0$ and V_1 is reductive.

Now we assume that V is rational. As before we need to show that the nilpotent radical N of the Lie algebra V_1 is zero. This time we use $\mathcal{C}(\sigma, \sigma)$ instead of $\mathcal{C}(\sigma, 1)$ and $\mathcal{C}(1, \sigma)$. In this case, the $S \in \Gamma$ maps $\mathcal{C}(\sigma, \sigma)$ to itself. The similar argument just applies. \square

Remark 5.4 *It is proved in [12] that if a vertex operator algebra is a strongly rational then weight one subspace is reductive. If one can prove that the rationality of V_0 from the rationality and σ -rationality of V then Theorem 5.3 will follow immediately. Unfortunately, none of these results have been established.*

The following result will be used in the next section.

Lemma 5.5 *Let V be a vertex operator superalgebra.*

(a) *If V is strongly rational, then any admissible V -module is completely reducible V_1 -module. This is also equivalent to saying the action of any Cartan subalgebra of Lie algebra V_1 is semisimple on any admissible V -modules.*

(b) *If V is strongly σ -rational, then any admissible σ -twisted V -module is completely reducible V_1 -module.*

(c) *If V is either strongly rational or strongly σ -rational, then any irreducible σ^i -twisted V -module is an semisimple V_1 -module for $i = 0, 1$.*

Proof. Since the proof of (b) is similar to (a) we only show (a) and (c) for strongly rational vertex operator superalgebra V . Let H be a Cartan subalgebra of V_1 . It is enough to show that H acts semisimply on any irreducible σ^i -twisted V -module for $i = 0, 1$. Since the homogeneous subspaces of an irreducible σ^i -twisted V -module is always finite dimensional there is a common eigenvector of H on the irreducible module. So it is enough to show that H acts on V semisimply.

First we show that for any $0 \neq u \in H$, h_0 is not nilpotent. Note that the restriction of the bilinear form $\langle \cdot, \cdot \rangle$ to H is nondegenerate. If u_0 for some $0 \neq u \in H$ is nilpotent, we can take $v \in H$ such that $\langle u, v \rangle = 1$. The proof of Theorem 5.3 then gives a contradiction.

We now prove that u_0 is semisimple on V . Since $\text{Aut}(V)$ is an algebraic group by Proposition 3.8 and $\{e^{tu_0} | t \in \mathbb{C}\}$ is one dimensional algebraic subgroup of $\text{Aut}(V)$ we see immediately see that $\{e^{tu_0} | t \in \mathbb{C}\}$ is isomorphic to the 1-dimensional multiplicative algebraic group \mathbb{C}_m as u_0 is not nilpotent (cf. [32]). This finishes the proof. \square

Now that V_1 is reductive, there are two extreme cases: V_1 is a semisimple Lie algebra and V_1 is abelian. The vertex operator subalgebra generated by V_1 will be extensively investigated in Section 6. We next study the rank of V_1 in the rest of this section. Let l be the rank of V_1 . That is, l is the dimension of a Cartan subalgebra H of V_1 . Similar to the case of vertex operator algebras in [12], l is closely related to the *effective central charge* \tilde{c} which is defined

as follows: Let $\{M^1, \dots, M^m\}$ be the irreducible σ -twisted V -modules up to isomorphism. Then there exist $\lambda_i \in \mathbb{C}$ such that $M^i = \sum_{n \in \frac{1}{2}\mathbb{Z}_+} M_{\lambda_i + n}^i$ with $M_{\lambda_i}^i \neq 0$. The λ_i is called the conformal weight of M^i . By Theorem 8.9 of [15], λ_i and the central charge c of V are rational numbers for all i . Define λ_{\min} to be the minimum of the conformal weights λ_i and set

$$\tilde{c} = c - 24\lambda_{\min}, \quad \tilde{\lambda}_i = \lambda_i - \lambda_{\min}.$$

Theorem 5.6 *Let V be strongly σ -rational. Then $l \leq \tilde{c}$*

Proof. Let H be a Cartan subalgebra of V_1 . Note that the component operators of the vertex operators $Y(u, z)$ on V for $u \in H$ form a Heisenberg Lie algebra. This amounts to saying that for $u, v \in H$ the following relations hold:

$$[u_m, v_n] = m\delta_{m, -n}\langle u, v \rangle. \quad (5.6)$$

In fact, these relations also hold true on any σ -twisted V -module M .

Consider $(g, h) = (1, 1)$. Let $F_i = F_{M^i}$ be as defined in (5.1). Then $F_i \in \mathcal{C}(1, 1)$. Recall that

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

is a modular form of weight $\frac{1}{2}$. Then

$$\eta(\tau)^{\tilde{c}} F_i(\tau) = q^{\tilde{\lambda}_i} \prod_{n=1}^{\infty} (1 - q^n)^{\tilde{c}} \sum_{n=0}^{\infty} \text{tr}_{M_{\lambda_i + \frac{n}{2}}} \phi(\sigma) q^{\frac{n}{2}}$$

is holomorphic in $\mathbf{H} \cup \{i\infty\}$. Now it follows from the transformation law for $\eta(\tau)$ and Theorem 5.2 that the m -tuple

$$(\eta(\tau)^{\tilde{c}} F_1(\tau), \dots, \eta(\tau)^{\tilde{c}} F_m(\tau))$$

is a holomorphic vector-valued modular form of weight $\tilde{c}/2$. So the Fourier coefficients of $\eta(\tau)^{\tilde{c}} F_i(\tau)$ have polynomial growth by Proposition 5.1.

The Stone-von-Neumann theorem provides us a somewhat different way to look more closely at $F_i(\tau)$. Namely, M^i has the following tensor decomposition:

$$M^i = M(1) \otimes_{\mathbb{C}} \Omega_{M^i}, \quad (5.7)$$

where $M(1) = \mathbb{C}[u_m | u \in H, m > 0]$ is the Heisenberg vertex operator algebra of rank l generated by H and $\Omega_{M^i} = \{w \in M^i | u_n w = 0 \text{ for } u \in H \text{ and } n > 0\}$. Then the trace function $F_i(\tau)$ corresponding to the decomposition (5.7) is equal to

$$q^{(l-c)/24} \eta(\tau)^{-l} \text{tr}_{\Omega_i} \phi(\sigma) q^{L(0)},$$

as $\text{tr}_{M(1)} \phi(\sigma) q^{L(0)} = q^{l/24} \eta(\tau)^{-l}$. Thus

$$\eta(\tau)^{\tilde{c}} F_i(\tau) = q^{(l-c)/24} \eta(\tau)^{\tilde{c}-l} \text{tr}_{\Omega_i} \phi(\sigma) q^{L(0)}. \quad (5.8)$$

We know that the Fourier coefficients of the left-hand side of (5.8) have polynomial growth. This forces the same is true on $\eta(\tau)^{\tilde{c}-l}$. Then one has $\tilde{c} - l \geq 0$, as $\eta(\tau)^s$ has exponential growth of Fourier coefficients whenever $s < 0$ (cf. [25]). \square

We now use Theorem 5.6 to do an estimation on the dimension of $V_{\frac{1}{2}}$.

Corollary 5.7 *Let V be strongly σ -rational. Then $\dim V_{\frac{1}{2}} \leq 2\tilde{c} + 1$.*

Proof. Let d be a nonnegative integer such that $2d \leq \dim V_{\frac{1}{2}} \leq 2d + 1$. Then there exists a unique (up to a constant) nondegenerate bilinear form satisfying (4.1). Note that the restriction of (\cdot, \cdot) to $V_{\frac{1}{2}}$ is still nondegenerate. So we can choose elements $b^i, b^{i*} \in V_{\frac{1}{2}}$ such that $(b^i, b^{j*}) = \delta_{ij}$ and $(b^i, b^j) = 0 = (b^{i*}, b^{j*})$ for all $1 \leq i, j \leq d$. Set $h^i = b_{-1}^i(b^i)^*_{-1}\mathbf{1}$ for $i = 1, \dots, d$. Then $h^i \in V_1$ and $h_1^i h^j = \delta_{ij}$, $h_0^i h^j = 0$ for $i, j \in \{1, \dots, d\}$. As a result, $\sum_{i=1}^d \mathbb{C}h^i \subset V_1$ is contained in a Cartan subalgebra of V_1 . By Theorem 5.7, $d \leq l \leq \tilde{c}$ and the proof is complete. \square

6. C_2 -confiniteness and Integrability

We continue our discussion on the weight 1 subspace V_1 . We will determine the vertex operator subalgebra $\langle V_1 \rangle$ of V generated by V_1 following the approach in [13]. It turns out that $\langle V_1 \rangle$ is isomorphic to $L_{\mathfrak{g}_1}(k_1, 0) \otimes \dots \otimes L_{\mathfrak{g}_s}(k_s, 0) \otimes M(1)$ where $V_1 = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s \oplus Z(V_1)$, \mathfrak{g}_i are simple, $k_i \geq 1$ are integers and $M(1)$ is the Heisenberg vertex operator algebra built up from $Z(V_1)$ (see below for the definition of $L_{\mathfrak{g}}(k, 0)$). Moreover, $\langle V_1 \rangle$ is contained in the rational vertex operator subalgebra $L_{\mathfrak{g}_1}(k_1, 0) \otimes \dots \otimes L_{\mathfrak{g}_s}(k_s, 0) \otimes V_L$ for some positive definite lattice $L \subset Z(V_1)$ satisfying $\text{rank}(L) = \dim Z(V_1)$.

Here we need to review the construction of untwisted affine Kac-Moody Lie algebras $\widehat{\mathfrak{g}}$ associated with simple Lie algebras \mathfrak{g} and relevant results from [22]. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} and Φ the corresponding root system. Fix a nondegenerate symmetric invariant bilinear form (\cdot, \cdot) on $\widehat{\mathfrak{g}}$ such that the square length of a long root is 2 where we have identified \mathfrak{h} with its dual via the bilinear form. Then the affine Kac-Moody algebra associated to \mathfrak{g} is given by

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

with the bracket relations

$$[u(m), v(n)] = [u, v](m+n) + m(u, v)\delta_{m+n,0}K, \quad [K, \widehat{\mathfrak{g}}] = 0 \quad (6.1)$$

for $u, v \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$ where $u(m) = u \otimes t^m$. Let $L(\lambda)$ be the irreducible \mathfrak{g} -module with highest weight $\lambda \in \mathfrak{h}$. Consider $L(\lambda)$ as a $\mathfrak{g} \otimes \mathbb{C}[t]$ -module with $\mathfrak{g} \otimes \mathbb{C}t[t]$ acting trivially and with K acting as the scalar $k \in \mathbb{C}$. Then the generalized Verma module

$$V(k, \lambda) = \text{Ind}_{\mathfrak{g}}^{\widehat{\mathfrak{g}}} L(\lambda) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K)} L(\lambda)$$

has the unique irreducible quotient $L(k, \lambda)$. It is well known that $L(k, \lambda)$ is integrable if, and only if, k is a nonnegative integer and λ is a dominant integral weight such that $(\lambda, \theta) \leq k$ where $\theta \in \Phi$ is the maximal root.

Let V be a VOSA of strong CFT type and $\langle \cdot, \cdot \rangle$ be the unique nondegenerate bilinear form satisfying $\langle \mathbf{1}, \mathbf{1} \rangle = -1$. Suppose that $\mathfrak{g} \subset V_1$ is a simple subalgebra. Then both bilinear forms (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$ on \mathfrak{g} are symmetric and invariant, so they must be proportional, that is,

$$\langle \cdot, \cdot \rangle = k(\cdot, \cdot) \quad \text{for some } k \in \mathbb{C}. \quad (6.2)$$

Then for any $u, v \in V_1$ and integers m, n one has

$$[u_m, v_n] = [u, v]_{m+n} + mu_1 v \delta_{m+n,0}.$$

Comparing this with (6.1) shows that the map

$$u(m) \rightarrow u_m \text{ for } u \in \mathfrak{g} \text{ and } m \in \mathbb{Z}$$

together with $K \rightarrow k$ gives rise to a representation of $\widehat{\mathfrak{g}}$ of level k .

Now we are going to state our main result related to C_2 -integrability, which has already been proved to be true in [13] for vertex operator algebras satisfying the C_2 -cofiniteness. But given a vertex operator superalgebra $V = V_0 \oplus V_1$ which satisfies the C_2 -cofinite condition, generally we can not prove that the even part V_0 also has such property. So in this sense, the following result sharpens the Theorem 3.1 of [13] although the idea is similar.

Theorem 6.1 *Let V be a simple vertex operator superalgebra which is C_2 -cofinite of strong CFT type, with $\mathfrak{g} \subset V_1$ a simple Lie subalgebra, k the level of V as $\widehat{\mathfrak{g}}$ -module, and the vertex operator subalgebra U of V generated by \mathfrak{g} . Then the following hold:*

- (a) *The restriction of $\langle \cdot, \cdot \rangle$ to \mathfrak{g} is nondegenerate,*
- (b) *$U \cong L(k, 0)$,*
- (c) *k is a positive integer,*
- (d) *V is an integrable $\widehat{\mathfrak{g}}$ -module.*

Proof. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} and let $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$ be the corresponding Cartan decomposition of \mathfrak{g} . Since \mathfrak{g} is generated by subalgebras isomorphic to $sl(2, \mathbb{C})$ it is good enough to show the theorem for $\mathfrak{g} = sl(2, \mathbb{C})$. Let $\{h, x, y\}$ be the standard basis of \mathfrak{g} . Then $(\alpha, \alpha) = 2$ and $k = \frac{(\alpha, \alpha)}{2}$ from this and equation (6.2).

Clearly, $U = \langle \mathfrak{g} \rangle$ is a quotient of $V(k, 0)$. So U is a $\widehat{\mathfrak{g}}$ -integrable module if and only if $U = L(k, 0)$ for some $k \in \mathbb{Z}_+$. This is also equivalent to the existence of a positive integer r such that

$$(x_{-1})^r \mathbf{1} = 0. \tag{6.3}$$

The proof of (6.3) is similar to the same result in [13] and we omit the proof. (b) then follows immediately. Also note that $\mathfrak{g} \subset U$, so U can not be a one-dimensional trivial module. Thus $k \neq 0$ and k must be a positive integer, proving (c) and (a). Since $L(k, 0)$ is rational (cf [7]), V is a direct sum of irreducible $L(k, 0)$ -modules, each of which is integrable as $\widehat{\mathfrak{g}}$ -module. Hence V is an integrable $\widehat{\mathfrak{g}}$ -module. This proves (d). \square

Next we consider a toral subalgebra of V_1 . Let V be strongly rational or strongly σ -rational and $\mathfrak{h} \subset V_1$ be a toral subalgebra such that the restriction of $\langle \cdot, \cdot \rangle$ to \mathfrak{h} remains nondegenerate. Notably, any Cartan subalgebra of V_1 automatically satisfies such condition.

Theorem 6.2 *Suppose that V is strongly rational or strongly σ -rational. Let $\mathfrak{h} \subset V_1$ be a toral subalgebra such that the restriction of $\langle \cdot, \cdot \rangle$ to \mathfrak{h} is nondegenerate. Then there exist a positive-definite even lattice $L \subset \mathfrak{h}$ with rank $\dim \mathfrak{h}$ and a vertex operator supersubalgebra U of V such that $\mathfrak{h} \subset U \cong V_L$.*

This Theorem has been proved in [13] (also see [32]) for vertex operator algebra. The same argument using Lemma 5.5 is also valid for vertex operator superalgebra.

We now assume that

$$V_1 = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s \oplus \mathfrak{h}$$

where \mathfrak{g}_i are simple Lie algebras and $Z(V_1) = \mathfrak{h}$. By Theorems 5.3, 6.1 and 6.2 we have (see [13] and [32]):

Corollary 6.3 *The V contains a strongly rational vertex operator subalgebra*

$$U = L_{\mathfrak{g}_1}(k_1, 0) \otimes \cdots \otimes L_{\mathfrak{g}_s}(k_s, 0) \otimes V_L$$

where and the commutant U^c of U in V is a vertex operator superalgebra such that $U_1^c = 0$.

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